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STABILIZATION OF THE UNSTABLE EQUILIBRIA OF CHARGES BY INTENSE MAGNETIC FIELDS†

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The problem of stabilizing unstable (by Earnshaw's theorem) equilibria of a free charge in an electrostatic field by adding a steady magnetic field is considered. The additional Lorentz force that thereby arises has a gyroscopic form. An example of the possibility of stabilization in a rigorous relativistic formulation of the problem is given. Criteria for the stabilization of unstable equilibria of linearized systems are obtained. The conditions for charge stability in intense magnetic fields are investigated and estimates of the stabilization probability are given. Some multidimensional analogues of these results are presented. In particular, the problem of gyroscopic stabilization when the matrix of the gyroscopic forces is degenerate is considered. Some extremal criteria of the stability of the equilibrium positions are given. © 1997 Elsevier Science Ltd. All rights reserved.

1. EARNSHAW'S THEOREM

We know [1], that the equilibrium of a free charge in any electrostatic field is always unstable (Earnshaw's theorem, 1839). Existing proofs are based on a consideration of the equations in variations (see, for example, [1]). However, one can easily give examples of electrostatic fields which allow of higher-order discrete symmetries, when the Taylor series of the potential energy begins with terms of any power not less than the third. Here, a first-approximation analysis may not produce any conclusions regarding the stability of the equilibrium. The first rigorous and complete proof of Earnshaw's theorem was given in [2]. It was pointed out in [3] that Earnshaw's theorem also holds in the relativistic case. Of course, linearization of the relativistic equations leads to the ordinary linear Newton's equations. However, as was mentioned above, these linear equations become unsuitable for degenerate equilibria. The proof of the instability of equilibrium uses the property of the harmonicity of the potential and Lyapunov's first method for strongly non-linear systems [4].

Earnshaw's theorem can be extended to pseudo-Riemannian spaces, which are more general than Minkowski space. Suppose M^4 is a pseudo-Riemannian space–time with $+---$. We consider a certain time-like geodesic and in a certain neighbourhood of this we introduce semigeodesic coordinates x_i $(0 \leqslant i \leqslant 3)$, $x_0 = ct$, in which the pseudo-Riemannian metric has the form

$$ds^2 = \mu c^2 dt^2 - \sum_{i,j \geqslant 1} g_{ij} dx_i dx_j$$

The coefficients μ and g_{ij} depend on $\mathbf{x} = (x_0, \ldots, x_3)$. This frame of reference is said to be static if μ and g depend only on the spatial coordinates x_1, x_2, x_3. In spaces with static frames of reference there are non-trivial steady electric fields (see, for example, [5]). The equations of motion of a charge e and mass m are obtained from the variational principle

$$\delta \int (-mc)ds + e\omega = 0$$

where ω is a 1-form in M^4, which specifies a 4-potential of the electromagnetic field. The world lines of the electron, parametrized by time, satisfy differential equations which generalize the well-known Poincaré–Minkowski equations. Time-like geodesic spaces M^4 correspond to positions of equilibrium. It turns out that all these equilibria are unstable. This generalized Earnshaw theorem is proved by the method described in [3].

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2. THE POSSIBILITY OF STABILIZING THE EQUILIBRIUM OF A CHARGE IN A MAGNETIC FIELD

Suppose now that M^4 is a Minkowski space. The motion of the charge in an electric field **E** and a magnetic field **H** is described by the relativistic equation

$$\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right)^{\cdot} = e\left(\mathbf{E}+\frac{1}{c}[\mathbf{v},\ \mathbf{H}]\right) \tag{2.1}$$

Here $\mathbf{v} = \dot{\mathbf{x}}$ is the charge velocity and c is the velocity of light.

We will consider a steady electromagnetic field (**E** and **H** are clearly independent of time). The field **E** is potential: $\mathbf{E} = -\operatorname{grad}\varphi$. The magnetic component of the Lorentz force is a gyroscopic force: its presence has no effect on the conservation of total energy

$$F = -mc(c^2 - v^2)^{1/2} + \varphi \tag{2.2}$$

If $\mathbf{H} = 0$, all the equilibria (the stationary points of the potential φ) are unstable.

We will give a simple example which describes the possibility of stabilizing unstable equilibria using a steady magnetic field. We will assume that the electric field **E** is produced by two similar charges Q, situated on the x_3 axis a distance R from the origin of coordinates O. The point O is then an unstable position of equilibrium. The potential of the electric field is equal to $\varphi_+ + \varphi_-$, where

$$\varphi_\pm = eQ[x_1^2 + x_2^2 + (R \mp x_3)^2]^{-1/2}$$

The expansion of the total energy (2.2) is a Maclaurin series has the form

$$F = m(v_1^2 + v_2^2 + v_3^2)/2 - eQ(x_1^2 + x_2^2 - 2x_3^2)/R^3 + \ldots$$

If $eQ > 0$ (which we will also assume later), the degree of instability (the Morse index of the function F at the critical point $\mathbf{x} = \mathbf{v} = 0$) is equal to two. If the charges e and Q have opposite signs, the degree of instability is odd (equal to unity) and by the Kelvin–Chetayev theorem gyroscopic stabilization is impossible.

We will introduce the magnetic field $\mathbf{H} = (0, 0, H)$, $H = \text{const}$, which, of course, satisfies Maxwell's equations. Since the kinetic energy and the electromagnetic field are invariant under rotations around the x_3 axis, Eqs (2.1) admit of a Noether integral

$$\Phi = \frac{mc(v_1x_2 - v_2x_1)}{(c^2 - v_1^2 - v_2^2 - v_3^2)^{1/2}} + \frac{eH}{2c}(x_1^2 + x_2^2) = m(v_1x_2 - v_2x_1) + \frac{eH}{2c}(x_1^2 + x_2^2) + \ldots$$

We will seek Lyapunov's function in the form of a bundle of integrals $F + \lambda\Phi$, where $\lambda = \text{const}$. Choosing λ from the condition for this integral to be a minimum we obtain the following rigorous condition for Lyapunov stability

$$H^2 > 8Qmc^2/(eR^3)$$

3. THE CONDITIONS FOR GYROSCOPIC STABILIZATION

We will investigate the problem of the stabilization of unstable equilibria of a charge by a magnetic field in the linear approximation. Letting, for convenience

$$e\varphi/m \to \varphi, \quad e\mathbf{H}/(mc) \to \mathbf{H}$$

the linearized equation (2.1) can be written in the form

$$\ddot{\mathbf{x}} = -\partial\varphi/\partial\mathbf{x} + [\dot{\mathbf{x}},\ \mathbf{H}]; \quad \varphi = (A\mathbf{x},\ \mathbf{x})/2, \quad \mathbf{H} = (H_1,\ H_2,\ H_3) \tag{3.1}$$

By a suitable orthogonal transformation the matrix A can be reduced to diagonal form:

$A = \operatorname{diag}(a_1, a_2, a_3)$. Since $\operatorname{div} \mathbf{E} = 0$, we have

$$a_1 + a_2 + a_3 = 0 \tag{3.2}$$

If $A \neq 0$, there is at least one negative number among the numbers a_1, a_2 and a_3. Then, the equilibrium $\mathbf{x} = 0$ will be unstable (by Lyapunov's theorem). This is Earnshaw's theorem in the non-degenerate case, when $A \neq 0$ [1].

Taking the above agreements into account, we can write Eqs (3.1) in the following explicit form

$$\ddot{x}_1 + H_2\dot{x}_3 - H_3\dot{x}_2 + a_1x_1 = 0, \ldots \tag{3.3}$$

The general linear equations of gyroscopic systems with three degrees of freedom can be reduced to the same form. A specific feature of the problem in question is satisfying (3.2).

We will write the characteristic equation of linear system (3.3), taking (3.2) into account, as follows:

$$f(\lambda^2) = \lambda^6 + \alpha\lambda^4 + \beta\lambda^2 + \gamma = 0$$
$$\alpha = H_1^2 + H_2^2 + H_3^2, \quad \beta = a_1a_2 + a_2a_3 + a_3a_1 + a_1H_1^2 + a_2H_2^2 + a_3H_3^2,$$
$$\gamma = a_1a_2a_3$$

The equilibrium $\mathbf{x} = 0$ is necessarily stable if the third-degree polynomial f has three different negative real roots. This condition is equivalent to the inequalities

$$\beta > 0, \quad 0 < \gamma < \alpha\beta$$
$$D = \alpha^2\beta^2 - 4\alpha^3\gamma + 18\alpha\beta\gamma - 4\beta^3 - 27\gamma^2 > 0 \tag{3.4}$$

The first two inequalities guarantee that the roots of the polynomial f lie in the left complex half-plane (the Hurwitz criterion). The condition $D \geqslant 0$ (where D is the discriminant of the polynomial f) is equivalent to the roots of the equation $f = 0$ being real, and if $D > 0$ they are all different.

Inequalities (3.4) can be represented in a simple geometric form. If $\gamma < 0$, the degree of instability is odd and gyroscopic stabilization is impossible. Hence, we will consider the case when $\gamma > 0$ and put $u = \alpha\gamma^{-1/3}$, $v = \beta\gamma^{-2/3}$. The conditions of stability (3.4) then take the form

$$v > 0, \quad uv > 1, \quad u^2v^2 - 4(u^3 + v^3) + 18uv - 27 > 0$$

The corresponding region in the plane of the parameters u, v is shown in Fig. 1 (it is shown hatched). Its boundary has one singular point $u = v = 3$, in the neighbourhood of which this curve looks like a semicubic parabola. Note that the condition for stability $\alpha\beta > \gamma$ is automatically satisfied by virtue of the condition that the discriminant D must be positive.

Note that the conditions for the stability of the equilibrium $\mathbf{x} = 0$ of a general gyroscopic system with three degrees of freedom (3.3) have the same form (3.4) except that the sum $a_1 + a_2 + a_3$ must be added to the parameter α.

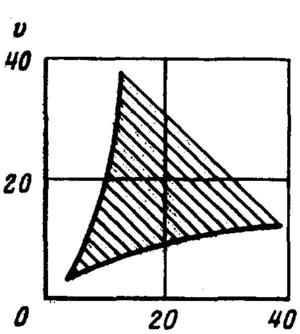

Fig. 1.

4. THE EXTREMAL CRITERION OF STABILITY

The condition for the stability of equilibrium can often be represented as the condition for an extremum of a certain function, which depends only on the position **x**. For example, if the potential energy $P = (A\mathbf{x}, \mathbf{x})/2$ has a strict minimum at the point $\mathbf{x} = 0$, the equilibrium is stable. Conversely, if the function $P + ([\mathbf{H}, \mathbf{x}], [\mathbf{H}, \mathbf{x}])/4$ reaches a maximum at $\mathbf{x} = 0$, the equilibrium is unstable [6]. Further results in this area can be found in the reviews [7, 8].

Proposition 1. If $\gamma > 0$, the equilibrium $\mathbf{x} = 0$ of system (3.3) is stable if and only if the quadratic form $(B\mathbf{x}, \mathbf{x})$, where

$$B = \begin{Vmatrix} \beta & \gamma^{1/2} & 2\alpha\gamma^{1/2} \\ \gamma^{1/2} & \alpha & -2\beta \\ 2\alpha\gamma^{1/2} & -2\beta & \alpha\beta + 27\gamma \end{Vmatrix} \tag{4.1}$$

has an absolute minimum at the point $\mathbf{x} = 0$.

In fact, by Sylvester's criterion, the quadratic form $(B\mathbf{x}, \mathbf{x})$ is positive definite when the diagonal minors of the matrix B are positive. It remains to verify that these conditions are identical with inequalities (3.4) when $\gamma > 0$.

Notes. 1. Conditions (3.4) guarantee that the roots of the characteristic polynomial f are pure imaginary and different. In the case of multiple roots the problem of the stability of equilibrium depends on the presence of Jordan cells. However, multiple roots are only encountered in the exceptional case when $D = 0$. Conditions (3.4) are the criterion for the strong stability of the equilibrium $\mathbf{x} = 0$: it remains stable for small changes in the coefficients of system (3.3).

2. It can be shown that there is a whole family of quadratic forms which satisfy Proposition 1. The symmetrical matrix (4.1) generates the simplest of these. However, it is difficult to give a clear mechanical interpretation to these forms. The possibility of extending Proposition 1 to the multidimensional case is discussed in Section 7.

5. INTENSE MAGNETIC FIELDS

If $\gamma = a_1a_2a_3 < 0$, the degree of instability is odd and gyroscopic stabilization is impossible. We will consider the case when $\gamma > 0$. We will fix the direction of the magnetic field **H** and increase its intensity $|\mathbf{H}|$.

Theorem 1. If the components of the magnetic field satisfy the inequality

$$\Sigma \equiv a_1H_1^2 + a_2H_2^2 + a_3H_3^2 > 0 \tag{5.1}$$

the equilibrium of the charge is stable for fairly large values of $|\mathbf{H}|$. If $\Sigma < 0$, the equilibrium is unstable for large values of $|\mathbf{H}|$.

In fact, if inequality (5.1) is satisfied, we have $\beta > 0$ for fairly large values of $|\mathbf{H}|$. Since $\alpha > 0$, the first two inequalities in (3.4) are satisfied. In order to prove inequality $D > 0$ it is sufficient to note that the term $\alpha^2\beta^2$ increases more rapidly than any other term as $|\mathbf{H}| \to \infty$. If $\Sigma < 0$, we have $\beta < 0$ for fairly large values of $|\mathbf{H}|$. So in this case the equilibrium is necessarily unstable.

Notes. 1. Inequality (5.1) is also the criterion for the stability of the position of equilibrium of a general linear system with three degrees of freedom on which large gyroscopic forces act.

2. If $\Sigma = 0$ and $A \neq 0$, the equilibrium of the charge is unstable. In fact, in this case (taking (3.2) into account) the coefficient $\beta = a_1a_2 - (a_1 + a_2)^2 < 0$.

We can give the stability condition (5.1) an interpretation in terms of geometric probability (see [9]). We fix the electric field and we assume that the necessary condition for gyroscopic stabilization is satisfied: the degree of instability is even. This indicates that there are two negative numbers and one positive number among the numbers a_1, a_2, a_3. Suppose, for example, that $a_1 < 0, a_2 < 0, a_3 > 0$. In three-dimensional Euclidean space $\mathbb{R}^3 = \{H_1, H_2, H_3\}$ the cone $\Sigma = 0$ intersects the unit sphere $S^2 = \{H^2_1 + H^2_2 + H^2_3 = 1\}$ along two ovals which divide it into three connected regions. The points S^2 of the two regions containing "poles"—points with coordinates $0, 0, \pm 1$, satisfy condition (5.1). The ratio of the sum of the areas of these regions to the area S^2 (it is equal to 4π) is the probability of stabilizing the unstable equilibrium by a randomly chosen magnetic field of greater intensity. The probability is equal to

$$p = \frac{1}{2\pi}\int_0^{2\pi} [1 - f(\varphi;\ a_1,\ a_2)]d\varphi, \quad f = \left(1 + \frac{a_1 + a_2}{a_1 \sin^2\varphi + a_2\cos^2\varphi}\right)^{-1/2}$$

and is expressed in terms of elliptic functions of a_1/a_2. Assuming $a_1/a_2 = \operatorname{tg}\alpha$ we obtain a function of one variable $\alpha \in [0, \pi/2]$. Its graph (Fig. 2) is symmetrical about the point $\alpha = \pi/4$, at which p takes the value $1-3^{-1/2} = 0.42\ldots$. In the interval $[\pi/4, \pi/2]$ the function $p(\alpha)$ increases, and its maximum value is $p(\pi/2) = 1/2$. Hence, the stabilization probability lies between the values 0.42... and 0.5. It is a minimum in the symmetrical case when $a_1 = a_2$, which was considered in Section 2.

6. SOME GENERALIZATIONS

We will consider the problem of the stability of the equilibrium of a linear system with n degrees of freedom, described by the equations

$$\mathbf{x}^{\cdot\cdot} + \Gamma\mathbf{x} + A\mathbf{x} = 0, \quad \mathbf{x} \in \mathbb{R}^n; \quad \Gamma^{\mathrm{T}} = -\Gamma, \quad A^{\mathrm{T}} = A, \quad \det A \neq 0 \tag{6.1}$$

We can assume the matrix A to be diagonal.

We replace Γ by $\mu\Gamma$ and assume that μ is a fairly large positive number. We will investigate what conditions the matrices A and Γ must satisfy for the equilibrium $\mathbf{x} = 0$ to be stable for large values of μ. The stability was established in [10, 11] on the assumption that the matrix A is negative (i.e. the potential energy $P(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})/2$ is negative-definite), while the matrix of the gyroscopic forces Γ is non-degenerate. Since $\Gamma^T = -\Gamma$ and $\det\Gamma \neq 0$, n is even. In this case the degree of instability is equal to n and hence is even. In this case the degree of instability is equal to n and hence is even. Estimates of the values of the parameter μ for which stability of the equilibrium $\mathbf{x} = 0$ occurs were given in [11, 12]. For odd n the matrix Γ is degenerate, and hence the results obtained in [10, 11] are inapplicable. When $n = 3$, Theorem 1 gives the stability criterion.

Suppose $\ker\Gamma$ is the kernel of the operator Γ, which is the set of all $\mathbf{x} \in \mathbb{R}^n$ such that $\Gamma\mathbf{x} = 0$. Clearly $\ker\Gamma$ is a linear subspace of $\mathbb{R}^n$. We can make the factor-space $\mathbb{R}^n/\ker\Gamma$ (see, for example, [13]) correspond to this; its elements are classes of vectors from $\mathbb{R}^n$, which differ by vectors from $\ker\Gamma$. We know [13], that $\mathbb{R}^n/\ker\Gamma$ has the structure of a vector space, and

$$\dim\ker\Gamma + \dim\mathbb{R}^n/\ker\Gamma = n, \quad \dim\mathbb{R}^n/\ker\Gamma = \operatorname{rank}\Gamma$$

Consequently, the dimension of the factor-space $\mathbb{R}^n/\ker\Gamma$1⁄θω is always even.

Suppose C is a symmetric matrix. We will compare the following quadratic form with it

$$(C\Gamma\mathbf{x},\ \Gamma\mathbf{x}) = -(\Gamma C\Gamma\mathbf{x},\ \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n \tag{6.2}$$

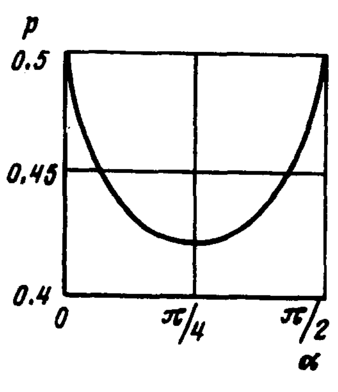

Fig. 2.

The value of this form does not change if we add to the vector $\mathbf{x}$ any vector of the subspace $\ker \Gamma$. The quadratic form (6.2) is of course correctly defined in the factor-space $\mathbb{R}^n/\ker \Gamma$.

Theorem 2. If the restriction on the potential energy $(A\mathbf{x}, \mathbf{x})/2$ in the subspace $\ker \Gamma$ is a positive-definite quadratic form, while the form $(A^{-1}\Gamma\mathbf{x}, \Gamma\mathbf{x}) = (A^{-1}\mathbf{z}, \mathbf{z})$ is negative-definite in $\mathbb{R}^n/\ker \Gamma$, the equilibrium $\mathbf{x} = 0$ is stable for fairly large values of the parameter μ.

We will consider the special case when $\det \Gamma \neq 0$. Then $\ker \Gamma = 0$, and the first condition of Theorem 2 can be omitted in view of its triviality. Suppose $\mathbf{x} = \Gamma^{-1}\mathbf{z}$. Then the form $(A^{-1}\Gamma\mathbf{x}, \Gamma\mathbf{x}) = (A^{-1}\mathbf{z}, \mathbf{z})$ will be negative-definite. Hence, it follows, in turn, that the potential energy $(A\mathbf{z}, \mathbf{z})/2$ will also be negative-definite. Hence, Theorem 2 contains the result derived in [11] as a special case (without estimates of the parameter μ).

Note. It is sufficient to verify the property that the quadratic form $(A^{-1}\Gamma\mathbf{x}, \Gamma\mathbf{x})$ is negative-definite in a certain subspace $\mathbb{R}^n$ with dimension n—dim $(\ker \Gamma)$, transverse to the space $\ker \Gamma$.

A further consequence that we can derive from Theorem 2 is inequality (5.1) as the sufficient condition for gyroscopic stabilization. When $n = 3$ the kernel of the operator $\Gamma \neq 0$ is one-dimensional and consists of vectors parallel to the vector (H_1, H_2, H_3). The value of the potential energy on this vector is $\Sigma a_k H_k^2/2$. Consequently, the first condition of Theorem 2 gives inequality (5.1). Since $\gamma = a_1 a_2 a_3 > 0$, then either $a_k > 0$ or two of these numbers are negative while one is positive. In the first case the equilibrium $\mathbf{x} = 0$ is stable by Kelvin's theorem. We will consider the second case; suppose, for example, $a_1 < 0, a_2 < 0, a_3 > 0$. Since $\Sigma a_k H_k^2 > 0$, we have $H_3 \neq 0$. Of course, we can take the $x_3 = 0$ plane as a two-dimensional subspace transverse to $\ker \Gamma$. The value of the quadratic form $(A^{-1}\Gamma\mathbf{x}, \Gamma\mathbf{x})$ on the vector $\mathbf{e}_1 = (1, 0, 0)$ from this plane is

$$H_3^2 / a_2 + H_2^2 / a_3 = (a_2 H_2^2 + a_3 H_3^2) / (a_2 a_3) < 0$$

since $a_2 H_2^2 + a_3 H_3^2 > -a_1 H_1^2 \geqslant 0$ and $a_2 a_3 < 0$. It can similarly be proved that this form is negative in any basis vector $\mathbf{e}_2 = (0, 1, 0)$. Hence, the second condition of Theorem 2 is satisfied.

Proof of Theorem 2. In addition to the energy integral

$$F = (\mathbf{x}^{\cdot}, \mathbf{x}^{\cdot})/2 + (A\mathbf{x}, \mathbf{x})/2$$

system (6.1) also allows of the integral

$$\Phi = (A^{-1}\mathbf{x}^{\cdot}, \mathbf{x}^{\cdot})/2 - (\Gamma A^{-1}\mathbf{x}^{\cdot}, \mathbf{x}) + ((E - \Gamma A^{-1}\Gamma)\mathbf{x}, \mathbf{x})/2$$

Here E is the identity matrix. We replace Γ by $\mu\Gamma$ and consider the quadratic integral

$$V = 2F - 2\Phi / \mu^{3/2} = (\mathbf{x}^{\cdot}, \mathbf{x}^{\cdot}) + (A\mathbf{x}, \mathbf{x}) - \mu^{1/2}(A^{-1}\Gamma\mathbf{x}, \Gamma\mathbf{x}) + O(\mu^{-1/2})$$

By the conditions of Theorem 2 the form $(A^{-1}\Gamma\mathbf{x}, \Gamma\mathbf{x})$ is non-positive and vanishes only in the subspace $\ker \Gamma$, where the form $(A\mathbf{x}, \mathbf{x})$ is positive-definite. Consequently, for fairly large values of μ the quadratic form V will be a positive-definite integral. By Lyapunov's theorem the state of equilibrium $\mathbf{x} = 0, \dot{\mathbf{x}} = 0)$ is stable.

We will indicate one more extension of condition (5.1) in the multidimensional case. Suppose $\gamma_{ij} = -\gamma_{ji}$ are the elements of the matrix of the gyroscopic forces Γ, $A = \mathrm{diag}(a_1, \ldots, a_n)$, $\alpha = \Sigma a_{i_1} a_{i_2} \ldots a_{i_{n-2}} \gamma_{i_{n-1} i_n}$, where the summation is carried out over all subscripts $i_1 < i_2 < \ldots < i_{n-2}$, not identical with the subscripts $i_{n-1} < i_n$. When $n = 3$, the quantity α is identical with the left-hand side of inequality (5.1).

Proposition 2. If $\alpha < 0$, the equilibrium $\mathbf{x} = 0$ is unstable for fairly large values of μ.

In fact, in the stable case all the coefficients of the characteristic polynomial must be positive. The coefficient of λ^2 differs from α by terms which depend only on a_k. Hence, if $\alpha < 0$, this coefficient becomes negative for sufficiently large μ.

Note. The inequality $\alpha > 0$ is the criterion for strong stability only when $n = 3$. For $n = 4$ we must add one more inequality to this condition, which relates the coefficients of the matrix of the gyroscopic forces Γ.

7. THE GENERAL FORM OF THE CONDITIONS OF STABILITY

Suppose

$$f(\lambda^2) = \lambda^{2n} + \alpha_1\lambda^{2n-2} + \ldots + \alpha_n \tag{7.1}$$

is the characteristic polynomial of the linear system (6.1) without multiple roots. The coefficients $\alpha_1, \ldots, \alpha_n$ are polynomials of a_k and of the elements of the matrix Γ. The presence of multiple roots is equivalent to the discriminant D of polynomial (7.1) vanishing.

The equilibrium $\mathbf{x} = 0$ is stable if and only if all the roots of the polynomial f of the nth degree are real negative numbers. Consequently, the necessary condition for stability is that the inequalities $\alpha_1 > 0, \ldots, \alpha_n > 0$ should be satisfied. We will assume these conditions to be satisfied.

Suppose $z_1, \ldots, z_n$ are simple roots of the equation $f(z) = 0$. We will put

$$s_k = z_1^k + z_2^k + \ldots + z_n^k, \quad s_0 = n$$

The numbers s_k are expressed in terms of the coefficients α_j by the following Newton formulae

$$s_m + s_{m-1}\alpha_1 + \ldots + s_1\alpha_{m-1} + m\alpha_m = 0, \quad m \leq n$$
$$s_m + s_{m-1}\alpha_1 + \ldots + s_{m-n}\alpha_n = 0, \quad m > n$$

Hence $s_1 = -\alpha_1, s_2 = \alpha^2{}_1 - 2\alpha_2, s_3 = -\alpha^3{}_1 + 3\alpha_1\alpha_2 - 3\alpha_3, \ldots$. We will introduce the nth order symmetric matrix

$$S = \left\| \begin{matrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{matrix} \right\|$$

and the quadratic form $\Phi(\mathbf{x}) = (S\mathbf{x}, \mathbf{x})$. Note that $D = \det S$.

Proposition 3. Suppose $\alpha_1 > 0, \ldots, \alpha_n > 0$ and $D \neq 0$. The equilibrium $\mathbf{x} = 0$ of system (6.1) is stable if and only if the function Φ has a strict minimum at the point $\mathbf{x} = 0$.

Proof. Suppose $D \neq 0$. Then the polynomial (7.1) does not have multiple roots. In this case, the criterion for all the roots of the polynomial f to be real is the condition for the matrix S to be positive-definite [14]. It remains to use the following result, which follows from Descarte's rule: the polynomial (7.1), without complex roots, has n negative roots if and only if all its coefficients are positive.

Notes. 1. Proposition 1 does not follow from Proposition 2, since the condition for the matrix (4.1) to be positive-definite includes the coefficients α and β being positive.

2. The characteristic polynomial of an autonomous Hamiltonian system, linearized in the neighbourhood of the equilibrium position, contains only even powers and hence has the form (7.1). Consequently, the conditions for the stability of the equilibria of Hamiltonian systems with n degrees of freedom reduces to $2n$–1 algebraic inequalities. Of course, not all of these are independent.

Example. When $n = 3$ we must add the following two conditions to the inequalities $\alpha > 0, \beta > 0, \gamma > 0$

$$\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = 2(\alpha^2 - 3\beta) > 0, \quad D = \det S > 0$$

Note that the first of these is a consequence of the second. Hence, the conditions for strong stability of the equilibrium position reduces to the already well-known four inequalities: $\alpha > 0, \beta > 0, \gamma > 0, D > 0$.

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REFERENCES

1. TAMM, I. Ye., *Principles of the Theory of Electricity*. Nauka, Moscow, 1966.
2. KOZLOV, V. V., On a Kelvin problem. *Prikl. Mat. Mekh.*, 1989, **53**, 1, 165–167.
3. VUJICIC, V. A. and KOZLOV, V. V., Lyapunov's problem on stability with respect to specified state functions. *Prikl. Mat. Mekh.*, 1991, **55**, 4, 555–559.
4. KOZLOV, V. V. and FURTA, S. D., Lyapunov's first method for strongly non-linear systems. *Prikl. Mat. Mekh.*, 1996, **60**, 1, 10–22.
5. EDDINGTON, A. S., *The Mathematical Theory of Relativity*. Cambridge University Press, Cambridge, 1954.
6. POZHARITSKII, G. K., The unsteady motion of conservative holonomic systems. *Prikl. Mat. Mekh.*, 1956, **20**, 3, 429–433.
7. KARAPETYAN, A. V. and RUMYANTSEV, V. V., The stability of conservative and dissipative systems. Advances in science and technology. *General Mechanics*, Vol. 6. Vsesoyuz. Inst. Nauch. Tekhn. Inf., Moscow, 1983.
8. BULATOVIC, R. M., The stability of linear potential gyroscopic systems when the potential energy has a maximum. *Prikl. Mat. Mekh.*, 1997, **61**, 3, 385–389.
9. KENDALL, M. G. and MORAN, P. A. P., *Geometric Probability*. Griffin, London, 1963.
10. MERKIN, D. R., *Gyroscopic Systems*. Nauka, Moscow, 1974.
11. LAKHADANOV, V. M., The stabilization of potential systems. *Prikl. Mat. Mekh.*, 1975, **39**, 1, 53–58.
12. KARAPETYAN, A. V., The problem of gyroscopic stabilization. *Teor. Primen. Meh.*, 1994, **20**, 89–93.
13. HALMOS, P., *Finite Dimensional Vector Spaces*. Ann. Math. Stud. No. 7. Princeton University Press, Princeton, NJ, 1942.
14. GRAVE, D. A., *Elements of Higher Algebra*. Izd. Imper. Univ. sv. Vladimira, Kiev, 1914.

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